An explanation of the "Precise Definition of a Limit" , Section 2.4

When one writes the formal $\varepsilon - \delta$ ("epsilon–delta" , although some call it the $\delta - \varepsilon$) definition of limit (Definition 2 on page 110):

" $\lim_{x \to a} f(x) = L$, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $0 < | x - a | < \delta$ then $| f(x) - L | < \varepsilon$ "

This means that if your choice of x (as an input value) is within δ units of a, then your output value $f(x)$ will be within ε units of L. Clearly (I hope), the closer you are to a —the smaller δ becomes— the closer you will get to L —the smaller ε becomes.

However, typically one tries to first determine how close to L one wants to end up; so therefore the thinking & work is actually done backwards: something like: If I know I want to end up being within ε units of L, then how close to a must I start? And so the 'work' is to determine a value of δ (almost always in terms of ε ; typically when you have an actual numerical value of ε , that will determine a numerical value of δ).

Example 2 and 3 in the text best show this algebraic process.

Ex. 2:
Prove that $\lim_{x \to 3} (4x - 5) = 7$
We know that if we use direct substitution, the limit is 7. But direct substitution is just a technique. How do we know it (or any other process) will *always* work? Thus we need a *proof* of the limit process. Definition 2 on page 110 provides the definition of "limit", and examples 2 & 3 show how it is applied.

The 'work' to 'prove the limit' is a two-part process:

1) doing some algebra to arrive at a value for δ in terms of ε (which typically means starting with $| f(x) - L | < \varepsilon$ and ending up with $| x - a | < \delta$

2) then "showing that this δ works" by choosing that value of δ (in terms of ε) start with $| x - a | < \delta$ (now replace the δ) and do the algebra (build up) to arrive at $| f(x) - L | < \varepsilon$

What happens if the value of δ (in terms of ε) also contains some expression in terms of x ? (see example 4) $\lim_{x \to 3} x^2 = 9$

Since $| x^2 - 9 | = | x - 3 | | x + 3 | < \varepsilon$, then $| x - 3 | < \frac{e}{|x + 3|} = \delta$ Now what?

When this happens, we now need to come up with a reasonable value of δ , and this will depend on the value of x . Therefore it is reasonable to assume that x is within 1 unit (an easy number to use) of 3 , or $x \ \varepsilon \ [2 , 4]$ (“ x is an element of the closed interval $[2 , 4]$ “) . Letting $x = 2$ means that $\delta = \frac{\varepsilon}{5}$ and letting $x = 4$ means that $\delta = \frac{\varepsilon}{7}$. Obviously $\frac{\varepsilon}{7}$ is smaller, so we take that fraction.

Therefore δ = the smaller of $\frac{e}{|x + 3|}$ or $\frac{\varepsilon}{7}$, depending on the choice of x . This is formally written as $\delta = \min (1 , \frac{e}{|x + 3|})$