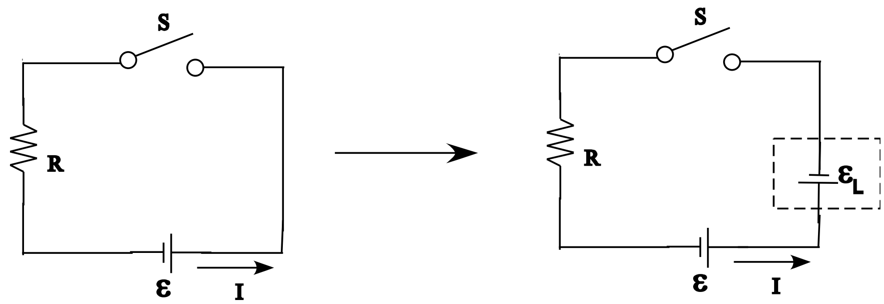
SELF-INDUCTANCE

Consider the following circuit:



A. When the switch is closed the current does not reach its max value of I = ε/R instantaneously. Faraday's Law can be used to explain why this occurs.
B. When the switch is closed the current increases and the magnetic flux throught the circuit increases.
C. This increase in flux induces an EMF such that it would cause an induced current that would oppose the increase in flux through the circuit.
D. Such induced EMF would have to be opposite to ε.
E. The result is a gradual increase of the current rather than an instantaneous increase.
F. This effect is called self-induction because the self-induced EMF arises from the circuit itself.

ε_L = self-induced EMF (back - emf)

The flux through the loop is proportional to the current in the loop:

(1) $$N\Phi_B = LI$$

Where the proportionality constant *L* is called the self-inductance.

$$\boxed{L = \frac{N\Phi_B}{I}}$$ Self-Inductance

Differentiating (1) $N\Phi_B = LI$ gives:

$$N\frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

$$\boxed{\varepsilon_L = -L\frac{dI}{dt}}$$ self-induced EMF

The negative sign is due to Lenz's Law which states that the induced EMF in a circuit opposes any change in the current in the circuit.

Properties of Inductance

1. Since $\frac{dI}{dt} = -\frac{\varepsilon_L}{L}$;

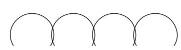
   a) The larger L, the smaller $\frac{dI}{dt}$, and the more slowly the current increases.

   b) The smaller L, the larger $\frac{dI}{dt}$, and the more faster the current increases.

   c) Thus, inductance is a measure of the opposition to changes in current.

2. The purpose of an inductor is to oppose any variations in the current through a circuit.

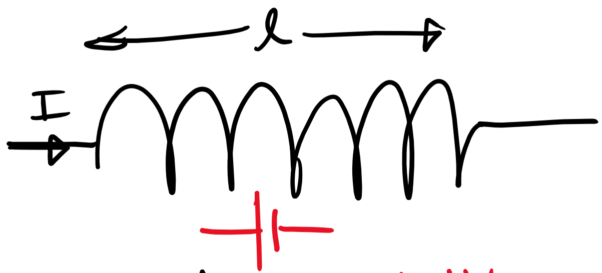
3. Circuits element that have large self-inductances are called inductors. Ex. Solenoids



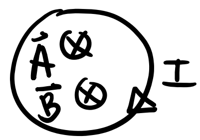
4. The current through an inductor cannot change instantaneously.

5. The SI unit of inductance is the Henry (H)
   1H = 1 V.s/A

Ex. Calculate the inductance of a solenoid with N = 300 turns, ℓ = 25,0 cm, and A = 4 x 10^{-2} m^2.



Back-view



$$L = \frac{N\Phi_B}{I} \qquad \varepsilon_L = -1.81V$$

$$\Phi_B = BA$$

$$= \mu_0 I n A$$

$$\Phi_B = \mu_0 I \frac{N}{\ell} A$$

$$L = \frac{N \mu_0 \cancel{I}}{\cancel{I}} \frac{N}{\ell} A$$

$$\boxed{L = \frac{\mu_0 N^2 A}{\ell}} = \left(4\pi \times 10^{-7} \frac{T\cdot m}{A}\right) \frac{(300)^2 (4\times10^{-2} m^2)}{0.25m}$$

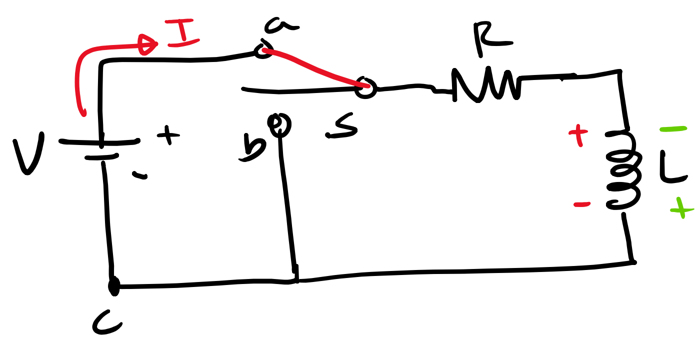
$$= \boxed{0.0181 H}$$

b) Calulate the self-induced EMF if the current is increasing at a rate of $100 \frac{A}{s}$.

$$\varepsilon_L = -L\frac{dI}{dt} = -(0.0181H)\left(100\frac{A}{s}\right)$$

$$\boxed{\varepsilon_L = -1.81 V}$$

RL Circuits



At $t=0$, S is set to "a". Applying the loop Rule:

$$\sum_{c\to c} \Delta V = 0$$

$$\boxed{V - IR - L\left(\frac{dI}{dt}\right) = 0}$$

At $t=0$ when "S" is set to "a", $I=0$:

$$V - L\frac{dI}{dt} = 0$$

$$\boxed{\frac{dI}{dt} = \frac{V}{L}} \quad \text{At } t=0$$

* the larger "L" the slower the current increases!

- As "I" increases and reaches its steady state value, then $\frac{dI}{dt}=0$:

$$V - IR - 0 = 0$$

$$\boxed{I = \frac{V}{R}} \quad \text{steady state value}$$

Let's find $I(t)$:

$$L\frac{dI}{dt} = V - IR$$

$$\frac{dI}{dt} = \frac{VR}{LR} - \frac{IR}{L}$$

$$\frac{dI}{dt} = -\frac{R}{L}\left(I - \frac{V}{R}\right)$$

$$\int_0^I \frac{dI}{I - \frac{V}{R}} = -\left(\frac{R}{L}\right)\int_0^t dt$$

let $u = I - \frac{V}{R}$, $du = dI$

$$\begin{cases} u_i = -\frac{V}{R} \\ u_f = I - \frac{V}{R} \end{cases}$$

$$\int_{-\frac{V}{R}}^{I-V/R} \frac{du}{u} = -\frac{R}{L} t' \Big|_0^t$$

$$\ln u \Big|_{-\frac{V}{R}}^{I-\frac{V}{R}} = -\frac{R}{L} t$$

$$\ln\left(I-\frac{V}{R}\right) - \ln\left(-\frac{V}{R}\right) = -\frac{R}{L} t$$

$$\ln\left(\frac{I-\frac{V}{R}}{-\frac{V}{R}}\right) = -\frac{R}{L} t$$

$$\ln\left(-\frac{IR}{V}+1\right) = -\left(\frac{R}{L}\right) t$$

$$1 - \frac{IR}{V} = e^{-\left(\frac{R}{L}\right)t}$$

$$\frac{IR}{V} = 1 - e^{-\left(\frac{R}{L}\right)t}$$

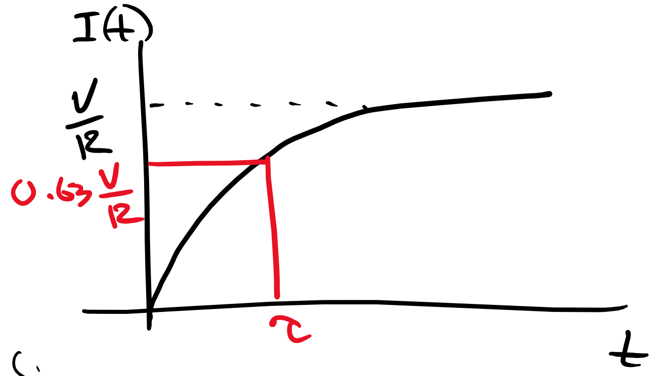
$$\boxed{I(t) = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)}$$

$\tau = \frac{L}{R}$ (time-constant)

$$I(t) = \frac{V}{R}\left(1 - e^{-t/\tau}\right)$$

$$I(\tau) = \frac{V}{R}\left(1 - e^{-1}\right)$$

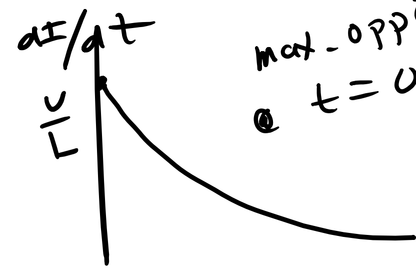
$$I(t) = 0.63\frac{V}{R} = 0.63 I$$



$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{V}{R} - \frac{V}{R} e^{-\left(\frac{R}{L}\right)}\right)$$

$$\frac{dI}{dt} = \left(\frac{V}{\cancel{R}}\right)\left(\frac{\cancel{R}}{L}\right) e^{-\left(\frac{R}{L}\right)}$$

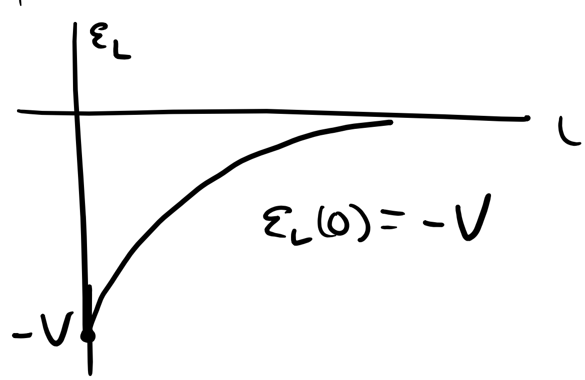
$$\frac{dI}{dt} = \left(\frac{V}{L}\right) e^{-\left(\frac{R}{L}\right)t}$$



max - opposite @ t = 0!

$$\varepsilon_L = -L\frac{dI}{dt}$$

$$\varepsilon_L = -V e^{-\left(\frac{R}{L}\right)t}$$



"S" is placed to "b" at t=0:

$$-IR - L\frac{dI}{dt} = 0$$

$$L\frac{dI}{dt} = -IR$$

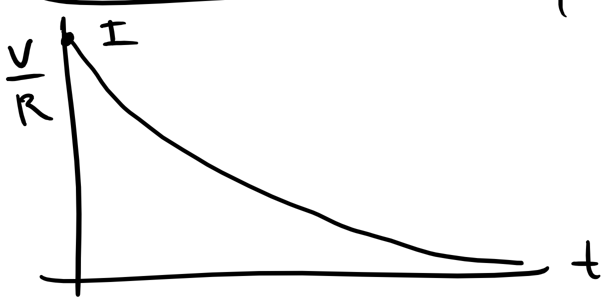
$$\int_{V/R}^{I}\frac{dI'}{I'} = -\left(\frac{R}{L}\right)\int_{0}^{t}dt'$$

$$\ln I'\Big|_{V/R}^{I} = -\left(\frac{R}{L}\right)t$$

$$\ln\frac{I}{V/R} = -\left(\frac{R}{L}\right)t$$

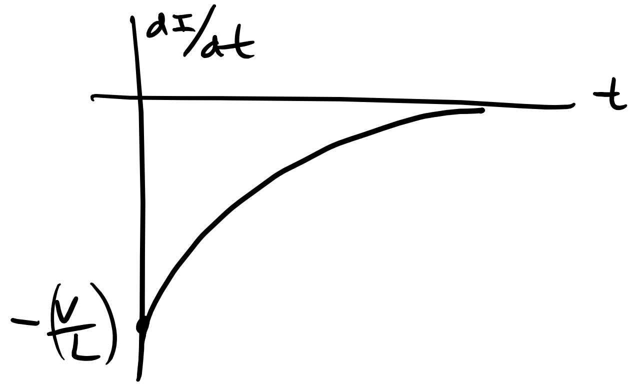
$$\frac{IR}{V} = e^{-(R/L)t}$$

$$\boxed{I = \frac{V}{R}e^{-(R/L)t}}$$



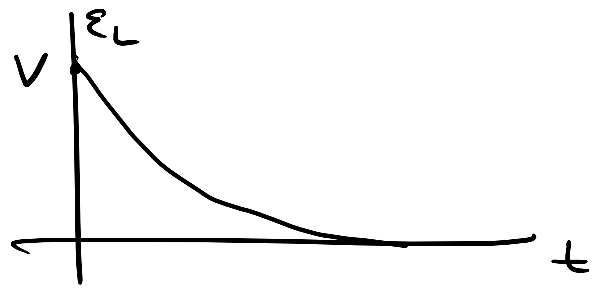
$$\frac{dI}{dt} = -\left(\frac{V}{\cancel{R}}\right)\left(\frac{\cancel{R}}{L}\right)e^{-t/\tau}$$

$$\boxed{\frac{dI}{dt} = -\left(\frac{V}{L}\right)e^{-t/\tau}}$$

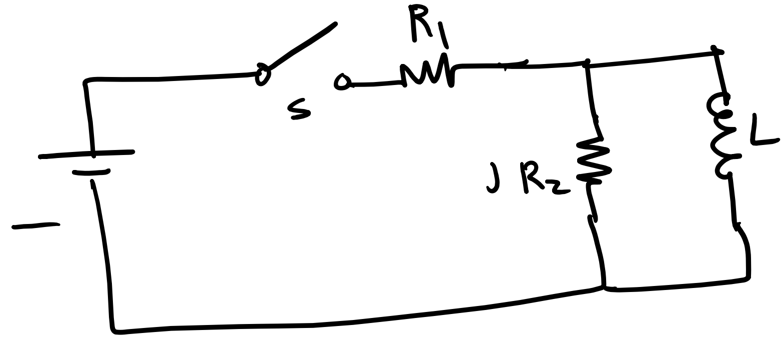


$$\mathcal{E}_L = -L\frac{dI}{dt}$$

$$\boxed{\mathcal{E}_L = V e^{-t/\tau}}$$

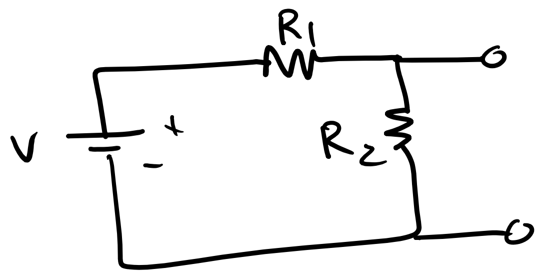


ex



a) Calculate the current in each element just after "S" is closed.

Since "I" cannot change inst. thru an inductor, then $I = 0$ thru "L" just after it's closed!

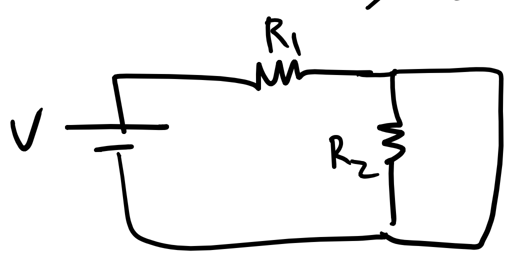


* Inductor acts like an open circuit!

$$\boxed{\begin{aligned} I_{R1} &= I_{R2} = \frac{V}{R_1 + R_2} \\ I_L &= 0 \end{aligned}}$$

b) Find "I" thru each element long time after "S" is closed.

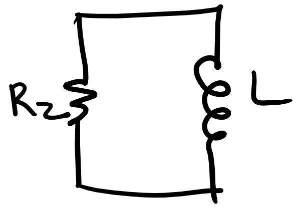
$$\mathcal{E}_L = -L \cancel{\frac{dI}{dt}}^{\,0} = 0$$ (reached steady-state current)



* Inductor acts like a short-circuit

$$\boxed{\begin{aligned} I_{R2} &= 0 \\ I_{R1} &= I_L = \frac{V}{R_1} \end{aligned}}$$

c) Long after "S" is closed, it is opened again.
Find "I" thru each element.



$\boxed{I_L = \frac{V}{R_1} = I_{R2}}$ ✓

$\boxed{I_{R1} = 0}$ ✓

$V_L = V_{R2} = I_{R2} R_2$

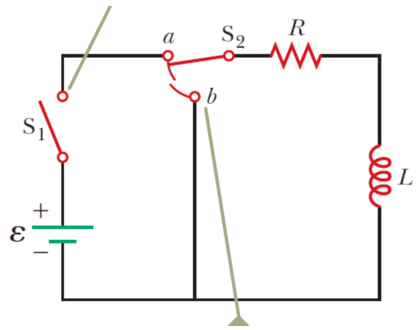
$\boxed{V_L = \frac{V}{R_1} R_2}$ ✓

If there is NO resistor connected across "L",
$R_2 = \infty$ and $V_L \to \infty$!
(very dangerous!)

MAGNETIC ENERGY IN AN INDUCTOR

In an RC-circuit we've learned that the work done by the battery, half ends up dissipated in the resistor and the other half ends up being stored in the capacitor. We were able to show that we can think of the energy as being stored in the E-field between the plates of the capacitor.

Similarly, in an RL-circuit half ends up dissipated in the resistor and the other half ends up being stored in the inductor. We will be able to show that we can think of the energy as being stored in the B-field inside the inductor.



$$V - IR - L\frac{dI}{dt} = 0$$
$$V = IR + L\frac{dI}{dt}$$

Multiplying both sides by I:

$$\boxed{IV = I^2R + LI\frac{dI}{dt}}$$

IV = power delivered by battery to circuit
I^2R = power delivered to resistor (energy dissipated in heat)
LIdI/dt = power delivered to inductor (energy stored in the B-field)

If U_B is the energy stored in the inductor at some time 't', then:

$$\frac{dU_B}{dt} = LI\frac{dI}{dt}$$

$$\int_0^{U_B} dU_B = L\int_0^{I} I' dI'$$

$$U_B = \frac{1}{2}LI'^2 \Big|_0^{I}$$

$$\boxed{U_B = \frac{1}{2}LI^2}$$ Magnetic PE stored in inductor

$$\approx \boxed{U_E = \frac{1}{2}CV^2}$$ EPE stored in capacitor

Magnetic Energy Density

Let's consider a solenoid:

$$\begin{cases} L = \mu_0 \dfrac{N^2}{l} A = \mu_0 \dfrac{N^2}{l^2} A l = \mu_0 n^2 V \\ B = \mu_0 I n \Rightarrow I = \dfrac{B}{\mu_0 n} \end{cases}$$

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2} \left(\cancel{\mu_0} \cancel{n^2} V\right) \left(\frac{B^2}{\cancel{\mu_0^2} \cancel{n^2}}\right)$$

$$U_B = \frac{1}{2} \frac{V B^2}{\mu_0}$$

$$\frac{U_B}{V} = \frac{1}{2} \frac{B^2}{\mu_0}$$

$$\boxed{u_B = \frac{1}{2} \frac{B^2}{\mu_0}}$$ Magnetic Energy Density $\approx \boxed{u_E = \frac{1}{2} \epsilon_0 E^2}$

MUTUAL INDUCTANCE

Often the magnetic flux through a circuit can vary due to the current changing in a nearby circuit. The EMF induced in a circuit this way is called *mutual inductance* because it is due to the interaction between the two coils.

Consider two closely wound coils of wire as shown below:

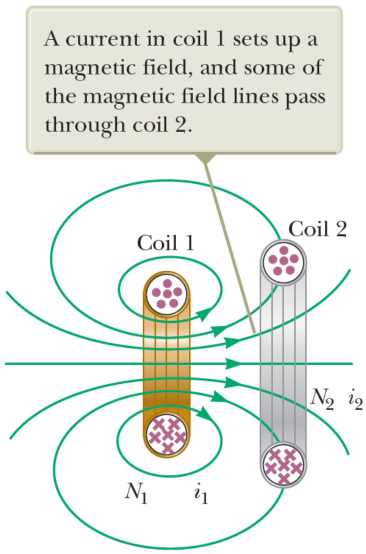


Figure 32.8 A cross-sectional view of two adjacent coils.

The flux through coil 2 is proportional to the current i_1 in coil 1:

$$N_2\Phi_B \propto i_1$$

(1) $N_2\Phi_B = M_{21}i_1$

$$\boxed{M_{21} = \frac{N_2\Phi_B}{i_1}}$$ Mutual Inductance

Differentiating Eq. (1):

$$N_2\frac{d\Phi_B}{dt} = M_{21}\frac{di_1}{dt}$$

$$\boxed{\varepsilon_2 = M_{21}\frac{di_1}{dt}}$$ Induced EMF in coil 2 due current changing in coil 1

If we now consider the current i_2 in the second oil changing with time:

$$M_{21} = \frac{N_2 \Phi_B}{i_1}$$

$$\varepsilon_1 = M_{12} \frac{di_2}{dt}$$

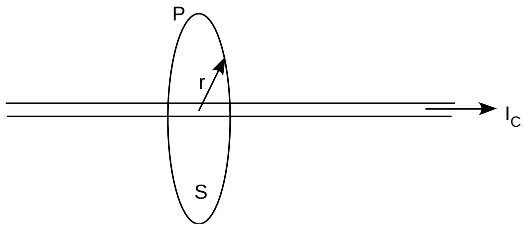
Although, not obvious:

$$M_{21} = M_{12}$$

The mutual inductance depends on the physical arrangement of both coils regardless of which one is causing the flux to change.

MAXWELL'S DISPLACEMENT CURRENT

A. Recall Ampere's Law



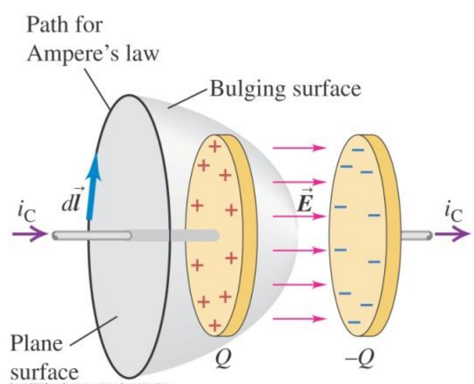
$$\oint \vec{B} \bullet d\vec{l} = \mu_o \, I_c$$

The line integral of $\vec{B} \bullet d\vec{l}$ around **any closed path** equals $\mu_o I_C$ where I_C is the total steady-state conduction current passing through **any surface** bounded by the closed path.

$$B 2\pi r = \mu_o I_c$$

$$B = \frac{\mu_o I_c}{2\pi r}$$

B. Maxwell's Difficulty
Consider a charging capacitor.



Consider the two surface – the plane surface and the bulging surface:

Plane Surface

$$\oint_{\substack{plane \\ surface}} \vec{B} \bullet d\vec{l} = \mu_o \, I_c$$

Bulging Surface

$$\oint_{\substack{bulging \\ surface}} \vec{B} \bullet d\vec{l} = 0 \quad \text{Thus, clearly a contradiction!!}$$

C. Maxwell's Solution

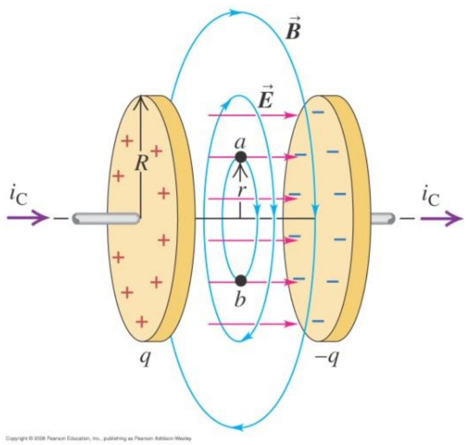
Maxwell solved this problem by postulating an additional term in Ampere's Law called Displacement Current.

$$\boxed{I_d = \varepsilon_o \frac{d\Phi_E}{dt}}$$ Displacement Current

$$\Phi_E = \int \vec{E} \bullet d\vec{A}$$

Where did this "Displacement Current" come from????

For the 'bulging surface' we can see that the E-field and thus the flux Φ_E are changing as capacitor is being charged. To find the rate at which they change (increase):



$$E = \frac{\sigma}{\varepsilon_o} = \frac{q}{A\varepsilon_o}$$
$$q = \varepsilon_o EA$$
$$I_c = \frac{dq}{dt} = \varepsilon_o \frac{d(EA)}{dt} = \varepsilon_o \frac{d\Phi_E}{dt} = I_d$$
$$Thus,$$
$$I_c = I_d$$

$$\boxed{\oint \vec{B} \bullet d\vec{l} = \mu_o (I_c + I_d) = \mu_o I_c + \mu_o \varepsilon_o \frac{d\Phi_E}{dt}}$$ Maxwell-Ampere's Law

Magnetic fields are produced both by conduction currents and by a changing electric flux.